A close look at quadratic equations

1. Introduction

A rather common topic in maths: how to best solve quadratic equations. Not really rocket science, but there are a few tricks to do it fast, systematically and minimize chances of errors.

CHOOSE THE SIMPLE METHODS FIRST TO SAVE TIME, DEPENDING ON THE PROBLEM AT HAND!

2. Sequential approach

Let us take the usual formula for a quadratic equation: $f(x) = ax^2 + bx + c = 0$.

1) First, if you can recognize one of the remarkable identities, for instance $\alpha^2 - \beta^2 = (\alpha - \beta)(\alpha + \beta)$, then you are immediately done. For the $\alpha^2 - \beta^2$ situation, there are two roots corresponding to $\alpha = \beta$ and $\alpha = -\beta$.

```
Example: f(x) = 4x^2 - 4x + 1
This is the same pattern as the identity \alpha^2 - 2 \alpha \beta + \beta^2 = (\alpha - \beta)^2
Therefore: f(x) = (2x - 1)^2 with a double root x = 1/2
```

2) When case 1 is not applicable, maybe it is possible to find an obvious root of the equation by direct observation. This often happens, for instance roots such as 1, 2, -1,... Let us call this obvious root x₁. Since x₁ is a root then you know that f(x) needs to be of the form (x - x₁)(ax +d) and the problem is almost solved. The constant terms in ax² + bx + c and (x - x₁)(ax +d) have to match, therefore c = - d x₁ and d = -c / x₁. As a consequence the second root is x₂ = -d/a = -c/(ax₁)

```
Example: g(x) = 2x^2 + 5x - 7 = 0
One obvious solution x = 1
g(x) = (x - 1) (2x + constant); the constant can be easily seen to be equal to 7.
Therefore: g(x) = (x - 1) (2x + 7) with roots 1 and -7/2
```

3) If the quick first and second methods are not applicable, the traditional "quadratic formula" method including the calculation of the discriminant $b^2 - 4ac$ can be applied; it works all the time, however at the expense of the required calculations.

An interesting and effective alternative to the quadratic formula is to transform f(x) into its so-called "canonical form" $a(x-\alpha)^2+\beta$. The method is also called "solving by completing the square":

For the x-terms to match between $ax^2 + bx + c$ and $a(x-\alpha)^2 + \beta = a(x^2 - 2\alpha x + \alpha^2) + \beta$, you need $b = -2 \alpha a$, therefore $\alpha = -b/2a$.

08-2023 Page 1 of 2

- Since ax^2 is replaced by $a(x-\alpha)^2$ in the expression $ax^2 + bx + c$, in order to get the constant terms correct, we need to subtract $a\alpha^2$:

$$f(x) = ax^2 + bx + c = a(x-\alpha)^2 - a\alpha^2 + c$$
; this gives $\beta = -a\alpha^2 + c$

- Conclusion: the canonical form of $ax^2 + bx + c$ is $a(x-\alpha)^2 + \beta$, with $\alpha = -b/2a$ and $\beta = -a\alpha^2 + c$
- The roots can be derived directly from the canonical form by solving $a(x-\alpha)^2 + \beta = 0$, which has real roots only if $\beta/a \le 0$.

```
Example: h(x) = x^2 - 4x - 1

h(x) can be re-written as h(x) = (x^2 - 4x + 4) - 4 - 1 = (x - 2)^2 - 5

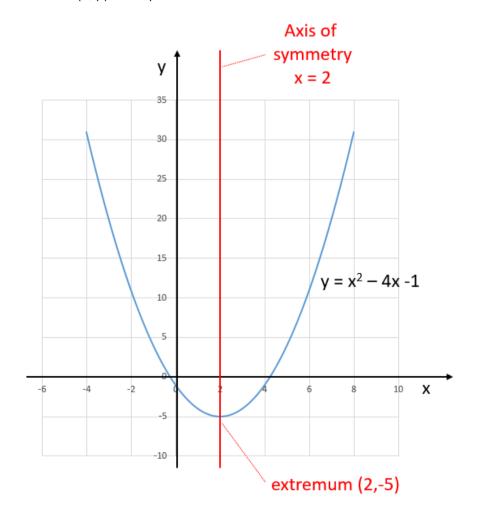
The roots are such that (x - 2)^2 = 5 i.e.

x - 2 = \pm \sqrt{5}

therefore the two roots are:

2 + \sqrt{5} : 2 - \sqrt{5}
```

As a bonus, the canonical form gives directly the axis of symmetry $(x = \alpha)$ of the parabolic curve corresponding to $f(x) = ax^2 + bx + c$ as well as the coordinates of the extremum (α, β) of the parabol.



08-2023 Page 2 of 2